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The Laplacian Spectral Excess Theorem for Distance-Regular Graphs*

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Abstract

The spectral excess theorem states that, in a regular graph Γ , the average excess, which is the mean of the numbers of vertices at maximum distance from a vertex, is bounded above by the spectral excess (a number that is computed by using the adjacency spectrum of Γ), and Γ is distance-regular if and only if equality holds. In this note we prove the corresponding result by using the Laplacian spectrum without requiring regularity of Γ .

Keywords: Distance-regular graph; Spectral excess theorem; Laplacian spectrum; Orthogonal polynomials.

1 Introduction

The spectral excess of a regular (connected) graph Γ is a number which can be computed from its (adjacency matrix) spectrum, whereas its average excess is the mean of the numbers of vertices at maximum distance from a vertex. The spectral excess theorem, due to Fiol and Garriga [15] states that Γ is distance-regular if and only if its spectral excess equals its average excess (see Van Dam [8] and Fiol, Garriga, and Gago [14] for short proofs). Since the paper [15] appeared, some attempts have been made to prove a version of the spectral excess theorem that does not require regularity of Γ (see Lee and Weng [19, 20] and Fiol [13]). The problem with these attempts is that the obtained equalities only lead to distance-regularity in some specific cases (graphs with extremal diameter, bipartite graphs, etc.), some of them already covered by the results in [15].

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In this note we show that the right approach to the spectral excess theorem for general graphs is to derive it from the Laplacian spectrum of the graph. This approach was motivated by the fact that a bound on the excess in terms of the Laplacian eigenvalues by the first author [7, Thm. 3.1] equals an expression for the excess in strongly distance-regular graphs by the second author and Garriga [16, Thm. 3.3], [12, Cor. 2.5]. In the following section we will recall the basic terminology and earlier results. Then the main result is derived in the last section.

2 Preliminaries

Let us first recall some basic notation and results on which our study is based. For more background on spectra of graphs, distance-regular graphs, and their characterizations, see [2, 3, 4, 6, 11, 17].

Throughout this paper, Γ denotes a (finite, simple, and connected) graph with vertex set V , order $n = |V|$, and diameter D . Its $(0, 1)$ -adjacency matrix is denoted by \mathbf{A} . The set of vertices at distance i from a given vertex $u \in V$ is denoted by $\Gamma_i(u)$, for $i = 0, 1, \dots, D$, and $k_i(u) = |\Gamma_i(u)|$. We abbreviate $k_1(u)$ by $k(u)$, the degree of vertex u . Also, the closed i -neighborhood of u is $N_i(u) = \Gamma_0(u) \cup \dots \cup \Gamma_i(u)$. Recall that, for every $i = 0, 1, \dots, D$, the distance matrix \mathbf{A}_i has entries $(\mathbf{A}_i)_{uv} = 1$ if $\text{dist}(u, v) = i$, and $(\mathbf{A}_i)_{uv} = 0$ otherwise. In particular, $\mathbf{A}_0 = \mathbf{I}$ and $\mathbf{A}_1 = \mathbf{A}$. Then, it is well-known that Γ is distance-regular if and only if there exist so-called distance polynomials p_0, \dots, p_D , with $\deg p_i = i$, such that $p_i(\mathbf{A}) = \mathbf{A}_i$ for every $i = 0, \dots, D$.

The Laplacian matrix of Γ is the matrix $\mathbf{L} = \mathbf{K} - \mathbf{A}$, where \mathbf{K} is the diagonal matrix with entries $K_{uu} = k(u)$, for $u \in V$. The (Laplacian) spectrum of Γ is $\text{sp } \Gamma = \text{sp } \mathbf{L} = \{\theta_0 (= 0)^{m_0}, \theta_1^{m_1}, \dots, \theta_d^{m_d}\}$, where $\theta_0 = 0 < \theta_1 < \dots < \theta_d$ are the distinct eigenvalues, and the superscripts stand for their multiplicities $m_i = m(\theta_i)$. In particular, since Γ is connected, $m_0 = 1$, and θ_0 has eigenvector \mathbf{j} , the all-1 vector. We emphasize that throughout this note, d will always denote the number of distinct eigenvalues minus one, and D will denote the diameter. Let \mathbf{F}_i , $i = 0, 1, \dots, d$, be the idempotents of \mathbf{L} , that is $\mathbf{F}_i = \frac{1}{\phi_i} \prod_{j \neq i} (\mathbf{L} - \theta_j \mathbf{I}) = \mathbf{U}_i \mathbf{U}_i^\top$, where $\phi_i = \prod_{j \neq i} (\theta_i - \theta_j)$, and \mathbf{U}_i is an $n \times m_i$ matrix having orthonormal eigenvectors of θ_i as columns. In particular, $\mathbf{F}_0 = \frac{1}{n} \mathbf{J}$, with \mathbf{J} being the all-1 matrix.

Laplacian predistance and Hoffman polynomials

Given a graph Γ with spectrum as above, the *Laplacian predistance polynomials* r_0, \dots, r_d , introduced analogously in [15] for the adjacency spectrum, are the orthogonal polynomials with respect to the scalar product

$$\langle p, q \rangle_\Gamma = \frac{1}{n} \text{tr}(p(\mathbf{L})q(\mathbf{L})) = \frac{1}{n} \sum_{i=0}^d m_i p(\theta_i) q(\theta_i), \quad p, q \in \mathbb{R}_d[x], \quad (1)$$

normalized in such a way that $\|r_i\|_\Gamma^2 = r_i(0)$. (This makes sense since it is known that, for any sequence of such orthogonal polynomials p_0, \dots, p_d , we always have $p_i(0) \neq 0$.) As every sequence of orthogonal polynomials, the r_i s satisfy a three-term recurrence of the form

$$xr_i = \beta_{i-1}r_{i-1} + \alpha_i r_i + \gamma_{i+1}r_{i+1}, \quad i = 0, \dots, d, \quad (2)$$

where $\beta_{-1} = \gamma_{d+1} = 0$, and $\beta_{i-1}\gamma_i > 0$ for $i = 1, \dots, d$. In fact, in our case it can be proved that the betas and gammas are negative, in a similar way as in [1, Lemma 2.3]. Also, similar as in the case of the adjacency predistance polynomials, it can be proved that $\alpha_i + \beta_i + \gamma_i = \theta_0 = 0$, $i = 0, \dots, d$, and $r_d(0) = n \left(\sum_{i=0}^d \frac{\phi_0^2}{m_i \phi_i^2} \right)^{-1}$, see [5].

Here we can also consider a *Hoffman-like polynomial* (see [18] for the case of the adjacency spectrum), defined as $H = \frac{n}{\phi_0} \prod_{i=1}^d (x - \theta_i)$, where we recall that $\phi_0 = \prod_{i=1}^d (-\theta_i)$. This polynomial satisfies $H(\mathbf{L}) = n\mathbf{F}_0 = \mathbf{J}$ (independently of whether Γ is regular or not), and $H = r_0 + r_1 + \dots + r_d$. The latter follows from the fact that $\langle H, r_i \rangle_\Gamma = \frac{1}{n} \text{tr}(H(\mathbf{L})r_i(\mathbf{L})) = \frac{1}{n} \text{tr}(r_i(0)J) = \|r_i\|_\Gamma^2$ for every $i = 0, \dots, d$. From $H(\mathbf{L}) = \mathbf{J}$ it follows that the diameter D is at most d .

3 The Laplacian spectral excess theorem

In this section we prove the main result, which can be considered as the spectral excess theorem for nonregular graphs. As in the short proofs of the (standard) spectral excess theorem, we prove the Laplacian version of such a result in two steps, that correspond to the lemmas below. Although the proofs of such lemmas are basically the same as in [14], we have detailed them in order to have this note more self-contained.

Lemma 1. *Let Γ be a graph with Laplacian matrix \mathbf{L} , predistance polynomials r_0, \dots, r_d , and distance matrices \mathbf{A}_i , $i = 0, \dots, d$. If $r_d(\mathbf{L}) = \mathbf{A}_d$ then, $r_i(\mathbf{L}) = \mathbf{A}_i$ for every $i = 0, 1, \dots, d$.*

Proof. We only show the case $i = d-1$, as the other cases are proved analogously. From the hypothesis and $H(\mathbf{L}) = \mathbf{J} = \sum_{i=0}^d \mathbf{A}_i$, we get that $r_0(\mathbf{L}) + \dots + r_{d-1}(\mathbf{L}) = \mathbf{A}_0 + \dots + \mathbf{A}_{d-1}$. We then distinguish three cases:

- If $\text{dist}(u, v) = d$, we clearly have $(r_{d-1}(\mathbf{L}))_{uv} = 0$.
- If $\text{dist}(u, v) = d-1$, the above gives $(r_{d-1}(\mathbf{L}))_{uv} = 1$.
- If $\text{dist}(u, v) \leq d-2$, the three-term recurrence for $i = d$ is $xr_d = \beta_{d-1}r_{d-1} + \alpha_d r_d$. Then, when applied to \mathbf{L} , we get that $\beta_{d-1}r_{d-1}(\mathbf{L}) = \mathbf{L}\mathbf{A}_d - \alpha_d \mathbf{A}_d$. But $(\mathbf{L}\mathbf{A}_d)_{uv} = \sum_{w \in V} (\mathbf{L})_{uw} (\mathbf{A}_d)_{wv} = \sum_{w \in N_1(u)} (\mathbf{L})_{uw} (\mathbf{A}_d)_{wv} = 0$ since $\text{dist}(w, v) \leq \text{dist}(u, v) + 1 \leq d-1$. Thus, $(r_{d-1}(\mathbf{L}))_{uv} = 0$ since $\beta_{d-1} \neq 0$.

Consequently, $r_{d-1}(\mathbf{L}) = \mathbf{A}_{d-1}$. □

Lemma 2. Let Γ be a graph with Laplacian predistance polynomial r_d . Let \bar{k}_d be the average over V of the numbers $k_d(u) = |\Gamma_d(u)|$. Then,

$$\bar{k}_d \leq r_d(0)$$

and, in case of equality, $r_d(\mathbf{L}) = \mathbf{A}_d$.

Proof. First, notice that $\langle r_d(\mathbf{L}), \mathbf{A}_d \rangle = \langle H(\mathbf{L}), \mathbf{A}_d \rangle = \langle \mathbf{J}, \mathbf{A}_d \rangle = \|\mathbf{A}_d\|^2 = \bar{k}_d$. Note that we use the inner product on matrices defined by $\langle \mathbf{M}, \mathbf{N} \rangle = \frac{1}{n} \text{tr}(\mathbf{M}\mathbf{N})$, so that $\langle p, q \rangle_\Gamma = \langle p(\mathbf{L}), q(\mathbf{L}) \rangle$ by (1). Also, by the Cauchy-Schwarz inequality, $|\langle r_d(\mathbf{L}), \mathbf{A}_d \rangle|^2 \leq \|r_d\|_\Gamma^2 \|\mathbf{A}_d\|^2 = r_d(0) \bar{k}_d$. Combining the above, the inequality holds. Moreover, in case of equality, $r_d(\mathbf{L}) = c\mathbf{A}_d$ for some constant c . Finally, we have that $c = 1$ because $\bar{k}_d = \langle r_d(\mathbf{L}), \mathbf{A}_d \rangle = \langle c\mathbf{A}_d, \mathbf{A}_d \rangle = c\bar{k}_d$ (and $\bar{k}_d = r_d(0) > 0$). \square

Now we are ready to give the spectral excess theorem for general graphs or, what we could call, the Laplacian spectral excess theorem.

Theorem 3. Let Γ be a graph on n vertices, with Laplacian spectrum $\{\theta_0(=0)^{m_0=1}, \theta_1^{m_1}, \dots, \theta_d^{m_d}\}$, and Laplacian predistance polynomial r_d . Let \bar{k}_d be the average over V of the numbers $k_d(u) = |\Gamma_d(u)|$. Then, Γ is distance-regular if and only if

$$\bar{k}_d = r_d(0) = n \left(\sum_{i=0}^d \frac{\phi_0^2}{m_i \phi_i^2} \right)^{-1},$$

where $\phi_i = \prod_{j \neq i} (\theta_i - \theta_j)$, $i = 0, \dots, d$.

Proof. For sufficiency, Lemmas 1 and 2 imply that $r_i(\mathbf{L}) = \mathbf{A}_i$ for every $i = 0, 1, \dots, d$. In particular, for $i = 1$, there exist some constants $\omega_1 \neq 0$ and ω_2 such that $\omega_1 \mathbf{L} + \omega_2 \mathbf{I} = \mathbf{A}$, which implies that $(\mathbf{L})_{uu} = -\omega_2/\omega_1$ for every $u \in V$. Then, Γ is regular with degree $k = -\omega_2/\omega_1$, and $\mathbf{L} = k\mathbf{I} - \mathbf{A}$. In turn, this assures the existence of the distance polynomials p_0, \dots, p_d of Γ , just take $p_i(x) = r_i(k - x)$ for $i = 0, \dots, d$, and hence Γ is distance-regular (with $D = d$). Necessity follows straightforwardly from $r_d(x) = p_d(k - x)$. \square

Let us illustrate this Laplacian spectral excess theorem in the case of graphs with three Laplacian eigenvalues, that is, the case $d = 2$. Such graphs have been studied in [10].

Note that for every d , we have that $r_0 = 1$, $r_1 = \frac{1}{\gamma_1}(x - \alpha_0)$, and that $\alpha_0 = \frac{1}{n} \text{tr} \mathbf{L} = \bar{k}$, the average degree. Moreover, it can be shown that $\gamma_1 = -1 + \bar{k} - \bar{k}^2/\bar{k}$, where $\bar{k}^2 = \frac{1}{n} \sum_{u \in V} k(u)^2$, using among others that $\frac{1}{n} \text{tr} \mathbf{L}^2 = \bar{k}^2 + \bar{k}$. Note that for a k -regular graph we thus have that $\alpha_0 = k$, and $\gamma_1 = -1$, so that $r_1 = k - x$, which corresponds to the fact that $\mathbf{A} = k\mathbf{I} - \mathbf{L}$.

For the case $d = 2$, the inequality $\bar{k}_2 \leq r_2(0)$ of Lemma 2 can be rewritten as $n - 1 - \bar{k} \leq H(0) - r_0(0) - r_1(0) = n - 1 + \frac{\alpha_0}{\gamma_1}$, which is equivalent to the inequality $\gamma_1 \leq -1$ (recall that γ_1 is negative), which in turn is equivalent to the inequality $\bar{k}^2 \geq \bar{k}^2$. This is of

course a standard inequality, and equality holds precisely when the graph is regular. Thus we may draw the (known) conclusion that a graph with three Laplacian eigenvalues is distance-regular (strongly regular in fact) precisely when it is regular.

A (non-regular) example with $D = d = 3$ is given by the path on four vertices, which has Laplacian spectrum $\{0, 2 - \sqrt{2}, 2, 2 + \sqrt{2}\}$. The betas, alphas, and gammas are as in below table.

i	0	1	2	3
β_i	$-3/2$	$-16/21$	$-7/10$	
α_i	$3/2$	$27/14$	$62/35$	$4/5$
γ_i		$-7/6$	$-15/14$	$-4/5$

The Laplacian predistance polynomials are

$$\begin{aligned}
 r_0 &= 1, \\
 r_1 &= -\frac{6}{7}x + \frac{9}{7}, \\
 r_2 &= \frac{4}{5}x^2 - \frac{96}{35}x + \frac{32}{35}, \\
 r_3 &= -x^3 + \frac{26}{5}x^2 - \frac{32}{5}x + \frac{4}{5}.
 \end{aligned}$$

Consequently, Lemma 2 gives the inequality $\bar{k}_3 \leq \frac{4}{5}$. Indeed, in this graph, we have that $\bar{k}_3 = \frac{1}{2}$. Note that this example has constant $k_2 = 1$, which reminds us of the version of the spectral excess theorem for regular graphs with $d = 3$ in [9] in terms of the number of vertices at distance two.

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